Section 11.7

② $f(x) = \sqrt{1+x}$ Maclaurin series $= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$

$f(0) = 1$

$f'(x) = \frac{1}{2\sqrt{1+x}}$ $f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$ $f''(0) = -\frac{1}{4}$

So

$m(x) = 1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2 + \cdots$

ⓑ By Taylor's theorem

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}|x|^{n+1}}{(n+1)!}$$

For $x = 1$

$$|f(1) - P_2(1)| \leq \frac{K_3}{3!}$$

K_3 is a bound on $|f'''(x)|$. By maple

$|f'''(x)| = \frac{3}{8} = K_3$

So $|f(1) - P_2(1)| \leq \frac{3/8}{3!} = \frac{1}{16}$

③ $f(x) = e^{2x}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$

The coefficient of x^{100} is when $k=100$ $\frac{(2x)^{100}}{100!} = \frac{2^{100}x^{100}}{100!}$

so the coefficient is $\boxed{\frac{2^{100}}{100!}}$

4 $f(x) = \frac{x}{1-x^3}$.

a) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots$

$\frac{x}{1-x^3} = x + x^4 + x^7 + x^{10} + \cdots = \sum_{k=0}^{\infty} x^{3k+1}$

b) The interval of convergence

$\lim_{k\to\infty} \frac{|x|^{3(k+1)+1}}{|x|^{3k+1}} = |x|^3 < 1$

$|x| < 1$

$-1 < x < 1$

At the endpoints

$x=1$ $\sum 1^{3k+1} = \sum 1$ divergent

$x=-1$ $\sum (-1)^{3k+1} = -1+1-1+1 \cdots$ divergent.

So the interval of convergence is $(-1, 1)$

c) Since $f(x) = \frac{x}{1-x^3} = x + x^4 + x^7 + x^{10} + \cdots$

$f'(x) = 1 + 4x^3 + 7x^6 + 10x^9 + \cdots$

$f''(x) = 4.3x^2 + 7.6x^5 + 10.9x^8 + \cdots$

OR differentiating $\left(\sum x^{3k+1}\right)''$ twice $= \sum_{k=1}^{\infty} (3k+1)(3k)x^{3k-1}$

d) Since $f(x) = \frac{x}{1-x^3} = x + x^4 + x^7 + x^{10} + \cdots$

$$\int_0^x f(t)\,dt = \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{8} + \cdots = \sum_{k=0}^{\infty} \frac{x^{3k+2}}{3k+2}$$

5) $f(x) = \frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right)$

a) We do know the Maclaurin series rep. for $\frac{1}{1+\frac{x}{2}}$

since

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1+\frac{x}{2}} = 1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \cdots$$

Hence

$$\frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{1}{2} - \frac{1}{2}\frac{x}{2} + \frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{1}{2}\left(\frac{x}{2}\right)^3 + \cdots$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2^{k+1}}$$

b) Recall $a_k = \frac{f^{k}(0)}{k!}$ Hence $f^{k}(0) = a_k \cdot k!$

$$f^{(259)}(0) = a_{259} \cdot 259!$$

$$a_{259} = \frac{(-1)^{259}}{2^{260}} = \frac{1}{2^{260}}$$

So $f^{(259)}(0) = \boxed{\frac{259!}{2^{260}}}$

7) $K_{n+1} = e^x$ Since $f^{n+1}(x) = e^x$ for all x, for all n

So we can take $K_{n+1} = e^x$.

so that $|f^{n+1}| \leq e^x = K_{n+1}$.

$$|f(x) - P_n(x)| = |e^x - P_n(x)| \leq \frac{K_{n+1}|x|^{n+1}}{(n+1)!}$$

$$\Rightarrow |e^x - P_n(x)| \leq \frac{e^x|x|^{n+1}}{(n+1)!}$$

as $\lim_{n\to\infty} P_n(x)$ becomes the Maclaurin series of e^x.

So

$$\lim_{n\to\infty} |e^x - P_n(x)| \leq \lim_{n\to\infty} \frac{e^x|x|^{n+1}}{(n+1)!} \to 0$$

So

$|e^x - m(x)| \to 0$ hence the error that $m(x)$ commits by approximating $e^x \to 0$.

So $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots = m(x)$.

9) a) If $|f^n(x)| \leq n$ then $K_{n+1} \leq n+1$

$$|f(x) - P_n(x)| \leq \frac{(n+1)|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n\to\infty.$$

b) Yes since

$$|f(x) - P_n(x)| \leq \frac{2^{n+1}|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n\to\infty \text{ for all } x.$$

12 $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{e^{-1/h^2} - 0}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h}$ $\left(\frac{0}{0}\right)$ L'hopital

Let $x = \frac{1}{h}$

$\lim_{x \to \infty} x e^{-x^2} = 0$

So $f'(0) = 0$

(b) Since all the derivatives will be 0 at 0 the maclourin series for f is the constant function 0.

(c) Radius of convergence is ∞.

(d) It converges to $f(x)$ when $x = 0$.